International Journal of Theoretical Physics, Vol. 24, No. 10, 1985

$SO(5, 1)$ Dynamical Symmetry for Electron Zitterbewegung

S. A. Bruce[1] and P. C. Minning[1]

Received April 15, 1985

Electron rest-frame internal canonical coordinates are reobtained by the free-particle Foldy–Wouthuysen transformation: Schrödinger "microscopic momentum", Barut–Bracken "microscopic coordinate", and the rest Hamiltonian, which describe Zitterbewegung in this frame. $SO(4, 1)$ Snyder space-time invariant quantization is considered in order to construct a dynamical group for Zitterbewegung. The electron's internal structure appears associated with its second-order self-energy process and governed by the 15-parameter dynamical group $SO(5, 1)$. This is a generalization of Barut–Bracken symmetry which describes Zitterbewegung as generated by an algebra of the rotation group $SO(5)$. This noncompact symmetry $SO(5, 1)$ permits a natural interpretation for the operators of its algebra and introduces a generalization to higher-dimensional fermionic representations.

1. INTRODUCTION

Earlier works (Saavedra and Utreras, 1981; Talukdar and Niyagi, 1982; Saavedra, 1981) have studied a possible generalization of quantum mechanics to high energies. The new results are required to be reducible to the known ones in the low-energy limit.

These studies use a one dimensional model in which the canonical commutator $[q, p]$ is generalized to high energies by

$$[q, p] = i\hbar + i(l/c)H \simeq i(l/c)H \tag{1}$$

where c is the velocity of light, H is the system Hamiltonian, and l is a constant with dimension length. Equation (1) implies a new uncertainty relation

$$\Delta q \Delta p \geq \frac{l}{2} cE, \qquad E = (c^2 p^2 + m^2 c^4)^{1/2} \tag{2}$$

[1]Department of Physics, Faculty of Sciences, P.O. Box 947, University of Concepción, Chile.

0020-7748/85/1000-0971$04.50/0

which introduces an interpretation for the parameter l: a particle cannot be localized better than $(\Delta q)_{\min} = l/2$.

We observe that relation (2) is found also in Dirac electron Zitterbewegung (Saavedra, 1981; Barut and Bracken, 1981; Schrödinger, 1930) when this movement is studied in the center of momentum ($\mathbf{p} = \mathbf{0}$) reference frame.

This suggests defining internal coordinates for the electron with the corresponding commutation relations obtained in such a way as to satisfy the uncertainty relation (2).

Barut and Bracken (1981) have introduced a global scheme that permits studying Zitterbewegung dynamics where the internal coordinates defined for the electron satisfy the algebra of the $SO(5)$ rotation group.

The objective of our present work is to generalize this result showing that it is possible to define in a natural way canonical coordinates for Zitterbewegung as generators of a more general global symmetry, governed by the dynamical group $SO(5, 1)$ (six-dimensional Lorentz group).

2. ELECTRON'S INTERNAL COORDINATES

Our first objective is to determine the internal dynamical variables (for the electron) that arise from the study of the nonrelativistic limit of Dirac theory. The well-known method of Foldy and Wouthuysen (1950) (FW) determines this limit to any order in v/c. The FW transformation is defined by means of the operator

$$F = \frac{1}{2}\left(\frac{2E}{mc^2 + E}\right)^{1/2}\left(I + \beta\frac{H_D}{E}\right) \tag{3}$$

with $F^\dagger F = \mathrm{I}$, where $H_D = c\boldsymbol{\alpha}\cdot\mathbf{p} + \beta mc^2$ is the Dirac Hamiltonian.

We obtain

$$H_{FW} = FH_D\mathrm{F}^\dagger = E\beta, \qquad E = (mc^2 + c^2p^2)^{1/2} \tag{4a}$$

If $\mathbf{x}$ is the electron position operator in the Dirac representation, then

$$\mathbf{x}_{FW} = F\mathbf{x}F^\dagger = \mathbf{x} - \frac{i\hbar c}{2E}\beta\boldsymbol{\alpha} + \frac{i\hbar c}{2Ep}\frac{\beta\boldsymbol{\alpha}\cdot\mathbf{pp}}{(E + mc^2)} - \frac{\hbar c^2}{2E}\frac{\boldsymbol{\Sigma} x\mathbf{p}}{(E + mc^2)} \tag{4b}$$

In the Dirac representation the particle's instantaneous velocity is given by

$$\mathbf{v} = \frac{i}{\hbar}[H_D, x] = c\boldsymbol{\alpha} \tag{4c}$$

Then

$$\mathbf{v}_{FW} = Fc\boldsymbol{\alpha}F^\dagger = \beta\frac{c^2}{E}\mathbf{p} + c\boldsymbol{\alpha} - \frac{c\boldsymbol{\alpha}\cdot\mathbf{p}}{(mc^2 + E)}\frac{c^2\mathbf{p}}{E} \tag{4d}$$

Now we study the nonrelativistic limit of equations (4). To order v/c we have

$$H_{\mathrm{FW}} \simeq mc^2\beta$$

$$\mathbf{x}_{\mathrm{FW}} \simeq \mathbf{x} + \delta\mathbf{x} = \mathbf{x} - \frac{i\hbar}{2mc}\beta\boldsymbol{\alpha} - \frac{\hbar}{4m^2c^2}\boldsymbol{\Sigma} x \mathbf{p} \tag{5}$$

$$\mathbf{v}_{\mathrm{FW}} \simeq \mathbf{v} + \delta\mathbf{v} = \frac{\beta}{m}\mathbf{p} + c\boldsymbol{\alpha}$$

The plane wave solutions in the FW representation can be written

$$\Psi_+^\uparrow = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad \Psi_+^\downarrow = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \quad \Psi_-^\uparrow = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \quad \Psi_-^\downarrow = \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \tag{6}$$

In the nonrelativistic limit we define the electron momentum in the FW representation as

$$\mathbf{p}_{\mathrm{FW}} = m\mathbf{v}_{\mathrm{FW}} \simeq \beta\mathbf{p} + mc\boldsymbol{\alpha} \tag{7}$$

This momentum consists of two terms: $\beta\mathbf{p}$ proportional to the usual momentum and $mc\boldsymbol{\alpha}$ which is an operator whose mean value is zero in states (6). Thus [when Ψ is any one of the spinors (6)] we have

$$\langle \mathbf{p}_{\mathrm{FW}} \rangle = (\Psi, \mathbf{p}_{\mathrm{FW}}\Psi) = \pm\mathbf{p} \tag{8}$$

as it should be.

If we put $\mathbf{x} = \mathbf{0}$ and $\mathbf{p} = \mathbf{0}$ (center of momentum reference frame) we obtain

$$H =: H_{\mathrm{FW}} = mc^2\beta$$

$$\mathbf{X} =: \mathbf{x}_{\mathrm{FW}} = -i(\hbar/mc)\beta\boldsymbol{\alpha} \tag{9}$$

$$\mathbf{P} =: \mathbf{p}_{\mathrm{FW}} = mc\boldsymbol{\alpha}$$

These same operators have been found from other considerations by other authors (Barut and Bracken, 1981; Schrödinger, 1930). In our case these quantities appear systematically as internal canonical coordinates for the electron. **X** and **P** are interpreted as the "microscopic position" (Barut and Bracken, 1981) and "microscopic momentum" (Schrödinger, 1930) of the electron, respectively.

The proof is straightforward that the operators (9) together with the spin components $S_k = (\hbar/4i)(\boldsymbol{\alpha} x \boldsymbol{\alpha})_k$ form an algebra of the five-dimensional rotation group $SO(5)$ (Barut and Bracken, 1981) (BB algebra):

$$\begin{aligned}
&[X_i, P_j] = -i\hbar\delta_{ij}\beta, && [X_i, X_j] = i(\hbar/m^2c^2)\varepsilon_{ijk}S_k \\
&[X_i, S_j] = i\hbar\varepsilon_{ijk}X_k, && [P_i, P_j] = i(4m^2c^2/\hbar)\varepsilon_{ijk}S_k \\
&[P_i, S_j] = i\hbar\varepsilon_{ijk}P_k, && [P_i, \beta] = -i(4m^2c^2/\hbar)X_i \\
&[X_i, \beta] = i(\hbar/m^2c^2)P_i, && [S_i, S_j] = i\hbar\varepsilon_{ijk}S_k \\
&[\beta, S_j] = 0
\end{aligned} \tag{10}$$

We point out that the first commutator of (10) can be written in the form (Saavedra, 1981)

$$[X_i, P_j] = -i\delta_{ij}(l/c)H \tag{11}$$

where $l = \hbar/mc$ is the electron Compton wavelength. Equation (11) implies the same uncertainty relations (2).

3. QUANTIZED SPACE-TIME

H. S. Snyder (1947) developed a theory that shows some commutation relations formally equal to those obtained in (10) for Zitterbewegung. He showed that Lorentz invariance does not require that a four-dimensional space-time be continuous. He demonstrated the existence of a discrete Lorentz-invariant space-time for which one must introduce a natural unit of length: Snyder considers the four-dimensional homogeneous hyperboloid

$$-\eta^2 = \eta_0^2 - \eta_1^2 - \eta_2^2 - \eta_3^2 - \eta_4^2 \tag{12}$$

where the η's are real coordinates of a four-dimensional space of constant curvature (five projective coordinates).

He defines the generalized space-time coordinate operators in addition to the generators of the ordinary Lorentz algebra as

$$\begin{aligned}
X_k &= ia\left(\eta_4\frac{\partial}{\partial\eta_k} - \eta_k\frac{\partial}{\partial\eta_4}\right) \\
X_0 &= ia\left(\eta_4\frac{\partial}{\partial\eta_0} + \eta_0\frac{\partial}{\partial\eta_4}\right) \\
L_k &= i\hbar\left(\eta_j\frac{\partial}{\partial\eta_i} - \eta_i\frac{\partial}{\partial\eta_j}\right) \\
M_k &= i\hbar\left(\eta_0\frac{\partial}{\partial\eta_k} + \eta_k\frac{\partial}{\partial\eta_0}\right)
\end{aligned} \tag{13}$$

where a is a constant with dimension length and i, j, k take the values 1, 2, 3 (cyclic).

The energy-momentum operators can be defined

$$p_\mu = (\hbar/a)(\eta_\mu/\eta_4), \qquad \mu = 0, 1, 2, 3 \tag{14}$$

The operators (13) and (14) close under commutation to form an algebra (S algebra) of the inhomogeneous five-dimensional Lorentz group $ISO(4, 1)$.

It can be shown that each operator X_k possesses a discrete spectrum with values ma where m is an integer. The operator X_0 has a continuous spectrum that extends from minus infinity to plus infinity. As can be seen in the definition (13), these operators will have an invariant spectrum only if η_4 is Lorentz invariant. This last condition is essential if we want the quadratic form (12) to be invariant.

The S algebra contains a total of 45 commutators. In particular we have

$$\begin{aligned} [X_i, X_j] &= i(a^2/\hbar)\varepsilon_{ijk}L_k \\ [X_i, L_j] &= i\hbar\varepsilon_{ijk}X_k \end{aligned} \tag{15}$$

closely similar to the corresponding commutators (10) if we consider L_k (Snyder angular momentum) to be the analogue of S_k (electron spin) and $a = \hbar/mc$.

However this result seems incomplete for the following reasons:

(a) Snyder postulates *a priori* an invariant coordinate η_4 without any specific physical reason.

(b) The S algebra is unable to reproduce the rest of the commutators (10).

(c) The Snyder algebra does not fulfill the Born reciprocity principle (Born, 1938) that establishes that the quantum canonical equations of movement should be invariant under the (canonical) transformation $\mathbf{X} \to \mathbf{p}$, $\mathbf{p} \to -\mathbf{X}$ (reciprocity transfomation).

Our program for the rest of this paper will be as follows:

(a) Introduce Snyder quantization as a consequence of quantum electrodynamics.

(b) Modify the theory so that it be invariant under the Born reciprocity transformation and obtain the equations of motion (10) for Zitterbewegung as part of a more general global symmetry.

(c) Show some fundamental properties of this new global symmetry.

4. FREE-ELECTRON MOTION EQUATIONS

Let us consider the motion of a free electron with 4-momentum p_μ ($\mu = 0, 1, 2, 3$).

We can form the invariant

$$p_\mu p^\mu = g_{\mu\nu} p^\mu p^\nu = m^2 c^2 \tag{16}$$

where m is the electron mass and

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = 1 \tag{17}$$

We consider the second-order self-energy process that corresponds to the virtual emission and reabsorption of a photon by the electron. (See Figure 1.) Of course, this process is the lowest-order interaction of the electron with its own quantized electromagnetic field.

The modified 4-momentum $p_\mu = \not{p}_\mu + k_\mu$ (Figure 1) now satisfies the condition

$$p_\mu p^\mu = g_{\mu\nu} p^\mu p^\nu = m^2 c^2 + 2\hbar(\not{p}_0 k_0 - \not{p} \cdot \mathbf{k}) \tag{18a}$$

Reordering terms we have

$$p_\theta^2 - p_1^2 - p_2^2 - p_3^2 - 2\hbar(\not{p}_0 k_0 - \mathbf{k} \cdot \not{p}) = m^2 c^2 \tag{18b}$$

This last equation can be written in the form

$$m_{\text{ob}}^2 - \delta m^2 = m^2 \tag{18c}$$

where m_{ob} is interpreted to be the "observable" mass, m the "bare" mass, and $\delta m^2 = 2\hbar(\not{p}_0 k_0 - \mathbf{k} \cdot \not{p}) \geq 0$ is the correction that must be added to m^2 so as to get m_{ob}^2.

Thus p^μ now fulfills the condition

$$p_\mu p^\mu = m_{\text{ob}}^2 c^2 \tag{18d}$$

For the electron, from (18b) it is possible to define a new Lorentz-invariant unobservable momentum coordinate p_4 by the equation

$$p_4 = \pm[2\hbar(\not{p}_0 k_0 - \not{p} \cdot \mathbf{k})]^{1/2} \tag{19}$$

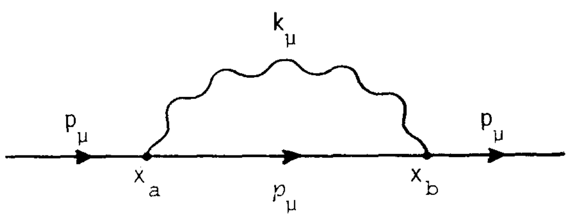


Fig. 1. Electron self-energy diagram. X_a and X_b should be considered to be any space-time points.

in such a way that

$$p_0^2 - p_1^2 - p_2^2 - p_3^2 - p_4^2 = p^2 = m^2c^2 \tag{20}$$

This quadratic form corresponds to a 4-hyperboloid H_4 embedded in a five-dimensional space R_5. Following Snyder [equations (13)] we define the infinitesimal generators of the homogeneous transformation group $SO(4, 1)$ that leaves (20) invariant[2]

$$\begin{aligned} X_k &= i\frac{\hbar}{p}\left(p_4\frac{\partial}{\partial p_k} - p_k\frac{\partial}{\partial p_4}\right) \\ X_0 &= -i\frac{\hbar}{p}\left(p_4\frac{\partial}{\partial p_0} + p_0\frac{\partial}{\partial p_4}\right) \\ L_k = M_{ij} &= i\hbar\left(p_j\frac{\partial}{\partial p_i} - p_i\frac{\partial}{\partial p_j}\right) \\ M_k = M_{k0} &= i\hbar\left(p_k\frac{\partial}{\partial p_0} + p_0\frac{\partial}{\partial p_k}\right) \end{aligned} \tag{21}$$

These operators satisfy

$$\begin{aligned} &[M_{\mu\nu}, M_{\phi\sigma}] = -i\hbar(g_{\mu\phi}M_{\nu\sigma} - g_{\nu\phi}M_{\mu\sigma} + g_{\mu\sigma}M_{\phi\nu} - g_{\nu\sigma}M_{\phi\mu}) \\ &[M_{\mu\nu}, p_0] = i\hbar(g_{\nu\sigma}p_\mu - g_{\mu\sigma}p_\nu) \\ &[M_{\mu\nu}, X_\sigma] = i\hbar(g_{\nu\sigma}X_\mu - g_{\mu\sigma}X_\nu) \\ &[M_{\mu\nu}, p_4] = 0, \qquad [p_\mu, p_\nu] = 0 \\ &[X_\mu, p_4] = i\frac{\hbar}{p}p_\mu, \qquad [X_\mu, X_\nu] = i\frac{h}{p}M_{\mu\nu} \end{aligned} \tag{22}$$

These are the commutation relations of the inhomogeneous five-dimensional Lorentz group. As in the case of Snyder, these equations are not invariant under the reciprocity transformation $X_\mu \to p_\mu$, $p_\mu \to -X_\mu$.

Now we observe more carefully the form that the electron spatial coordinates take in (21). X_k consists of two terms: one proportional to the ordinary coordinate $i\hbar(\partial/\partial p_k)$ and the other proportional to a spatial coordinate $x_4 = i\hbar(\partial/\partial p_4)$ according to the quadratic form (20). With this it is possible to define a quadratic invariant form similar to (20) but now in the electron spatial coordinates. That is, we can define the hyperboloid H_4 in R_5:

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 = r^2 \tag{23}$$

where x_4 is a Lorentz invariant and r is a constant with dimension length.

[2]Note that X_0 in (13) and X_0 *defined* by (21) have opposite signs.

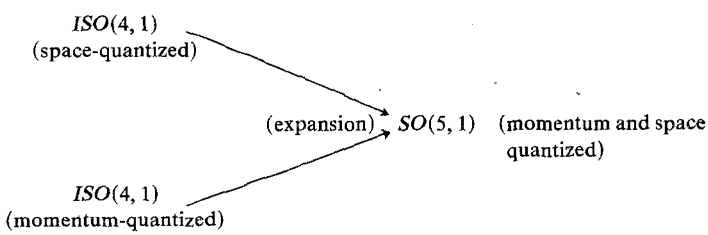
Consequently we can construct the generators of the homogeneous invariance group of (23):

$$
\begin{aligned}
P_k &= i\frac{\hbar}{r}\left(x_k\frac{\partial}{\partial x_4} - x_4\frac{\partial}{\partial x_k}\right)\\
P_0 &= i\frac{\hbar}{r}\left(x_0\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_0}\right)\\
L_k = M_{ij} &= i\hbar\left(x_j\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial x_j}\right)\\
M_k = M_{k0} &= i\hbar\left(x_k\frac{\partial}{\partial x_0} + x_0\frac{\partial}{\partial x_k}\right)
\end{aligned}
\tag{24}
$$

where now P_μ ($\mu = 0, 1, 2, 3$) is the generalized quantized 4-momentum and, for simplicity, we have used the same letters L_k and M_k as in (21) to designate the angular momentum operators and the "boosts" of the algebra, respectively. It is easy to verify that (22) are invariant interchanging $P_\mu \leftrightarrow -X_\mu$, $x_\mu \leftrightarrow p_\mu$, and $p \leftrightarrow r$.

Schemes (21) and (24) describe the same particle. However they are not compatible. For example x_μ and X_μ fulfill distinct commutation relations. Besides, neither scheme satisfies the Born reciprocity condition.

In order to solve these difficulties, it is most appropriate to think of X_μ and P_μ as generators of the same unique algebra. This can be done by means of expansion (Gilmore, 1974) of versions (21) and (24) of the (22) $ISO(4, 1)$ algebra into a homogeneous algebra of $SO(5, 1)$ (six-dimensional Lorentz group):



(25)

The generators of this algebra are then X_μ, P_μ, L_k, M_k, and H, where H is the Hermitian operator that we must add to complete an algebra (of 15 parameters) of $SO(5, 1)$. The possibility of generalizing the S algebra to a (no-compact) algebra of $SO(5, 1)$ was originally proposed by C. N. Yang (1947) in a comment made by him on a paper by Snyder.

The generators of this new global algebra (Z algebra) now act upon a universe of abstract coordinates that can be neither x space nor p space.

This new Lorentz-invariant universe can be represented by coordinates ν_a; $a = 0, \ldots, 5$ by means of the hyperboloid H_5 in R_6:

$$\nu_0^2 - \sum_{a=1}^{5} \nu_a^2 = \nu^2 \tag{26}$$

where ν_4 and ν_5 are Lorentz-invariant coordinates and ν is the radius of curvature of this universe.

Then the generators of the Z algebra can be written in the form

$$\begin{aligned} &X_\mu = (\hbar r/\nu) Z_{4\mu}, \qquad P_\mu = (\hbar p/\nu) Z_{\mu 5} \\ &L_k = \hbar Z_{ij}, \qquad M_k = \hbar Z_{j0} \\ &H = (\hbar \varepsilon/\nu) Z_{54} \end{aligned} \tag{27}$$

where

$$\begin{aligned} &Z_{ab} = i(\nu_a \partial_b - \nu_b \partial_a) \\ &Z_{0a} = i(\nu_0 \partial_a + \nu_a \partial_0), \qquad a, b = 1, \ldots, 5 \end{aligned} \tag{28}$$

$\nu = rp$ and ε is a quantity with the dimensions of energy. This form of writing the Z-algebra operators permits reobtaining, by contraction of $SO(5, 1)$, the original $ISO(4, 1)$ operators in two possible versions (21) and (24) ($\nu_5 = \nu = rp$, $r \to \infty$ and $\nu_4 = \nu = rp$, $p \to \infty$, respectively).

The operators (27) satisfy the invariant algebra

$$\begin{aligned} &[M_{\mu\nu}, M_{\phi\sigma}] = -i\hbar(g_{\mu\phi} M_{\nu\sigma} - g_{\nu\phi} M_{\mu\sigma} + g_{\mu\sigma} M_{\phi\nu} - g_{\nu\sigma} M_{\phi\mu} \\ &[M_{\mu\nu}, P_\sigma] = i\hbar(g_{\nu\sigma} P_\mu - g_{\mu\sigma} P_\nu) \\ &[M_{\mu\nu}, X_\sigma] = i\hbar(g_{\nu\sigma} X_\mu - g_{\mu\sigma} X_\nu) \\ &[M_{\mu\nu}, H] = 0 \\ &[X_\mu, H] = ih(\varepsilon/p^2) P_\mu, \qquad [P_\mu, H] = i\hbar(\varepsilon/r^2) X_\mu \\ &[X_\mu, X_\nu] = -i(\hbar/p^2) M_{\mu\nu}, \qquad [P_\mu, P_\nu] = -i(\hbar/r^2) M_{\mu\nu} \\ &[X_\mu, P_\nu] = i g_{\mu\nu} (\hbar/\varepsilon) H \end{aligned} \tag{29}$$

5. CONNECTION WITH ELECTRON ZITTERBEWEGUNG—FINAL COMMENTS

We observe that the BB-$SO(5)$ algebra (10) is contained in the $SO(5, 1)$ Z algebra (29) if we substitute $\nu = \hbar/2$, $r = \hbar/2mc$, $p = mc$, and $\varepsilon = mc^2$. This can be seen more clearly if we represent the Z algebra by means of the 4×4 γ matrices. The corresponding operators then are[3]

$$\begin{aligned} X_0 &= i(\hbar/2mc)\gamma_5, & X_j &= -(\hbar/2mc)\gamma_j \\ P_0 &= mc\gamma_0\gamma_5, & P_j &= imc\gamma_0\gamma_j \\ L_k &= i(\hbar/2)\gamma_j\gamma_i, & M_k &= (\hbar/2)\gamma_k\gamma_5 \\ H &= mc^2\gamma_0 \end{aligned} \tag{30}$$

where L_k corresponds to the electron spin S_k and H is its rest energy.

This representation of the Z algebra then corresponds to a generalization of the algebra of BB-$SO(5)\subset Z$-$SO(5, 1)$.

The above shows that Zitterbewegung is a consequence of the second-order self-energy process. This permits the construction of an invariant global algebra and the deduction of the commutation relations that generalize the BB algebra.

We observe that X_μ and P_μ in the Z algebra (29) satisfy harmonic oscilator equations:

$$\begin{aligned} \ddot{X}_\mu + \omega^2 X_\mu &= 0 \\ \ddot{P}_\mu + \omega^2 P_\mu &= 0 \end{aligned} \tag{31}$$

where $\omega = 2mc^2/\hbar$ and $\dot{X}_\mu = (i/\hbar)[H, X_\mu]$.

Defining the raising and lowering operators

$$\begin{aligned} a_\mu^\dagger &= (m\omega/2\hbar)^{1/2} X_\mu - i(m\hbar\omega/2)^{-1/2} P_\mu \\ a_\mu &= (m/2\hbar)^{1/2} X_\mu + i(m\hbar\omega/2)^{-1/2} P_\mu \end{aligned} \tag{32}$$

we can show that

$$g_{\mu\nu} H = (\hbar\omega/2)[a_\mu^\dagger, a_\nu] \tag{33}$$

The spectrum of the Hamiltonian H can be found using standard techniques. One finds that this energy spectrum consists of two states $\pm mc^2$, and that the ladder operators (32) change the sign of the energy. In a hole theory this shows that the Dirac equation describes a composite system consisting of two states: particle and antiparticle (Saavedra, 1981). One also shows that the corresponding operators $N_\mu = a_\mu^\dagger a_\mu$ (here we do not sum over μ) have only two eigenvalues $n = 1, 0$ (fermionic oscillator) corresponding to the particle and antiparticle, respectively.

[3]We use the Pauli metric: γ_μ are Hermitian matrices: $\gamma_0 = \beta$, $\gamma_k = i\beta\alpha_k$, $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$, and $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}$.

The generalization to higher dimensions follows immediately by means of the irreducible unitary representations of $SO(5,1)$ in (28) with the resulting associated energy spectrum (mass spectrum) for H.

ACKNOWLEDGMENTS

The authors specially wish to thank Dr. Igor Saavedra for several suggestions during the development of the present research and Dr. Patricio Cordero for his permanent interest and fruitful discussions about the subject of this work.

REFERENCES

Barut, A. O., and Bracken, A. J. (1981). *Physical Review D*, **23**, 2354.
Born, M. (1938). *Proceedings of the Royal Society of London Series A*, **165**, 291.
Foldy, L. L., and Wouthuysen, S. A. (1950). *Physical Review*, **78**, 29.
Gilmore, R. (1974). *Lie Groups, Lie Algebras, and Some of Their Applications*. Wiley, New York.
Saavedra, I. (1981). *Quantum Theory and the Structures of Time and Space*, Vol. 4, L. Castell, M. Driescher, and C. F. von Weizsacker, eds. Carl Hauser Verlag, Munich.
Saavedra, I., and Utreras, C. (1981). *Physics Letters*, **98B**, 74.
Schrödinger, E. (1930). *Sitzungsberichte der Preussischen Akademie der Wissenschaften Physikalisch-Mathematische Klasse*, **24**, 418.
Snyder, H. S. (1947). *Physical Review*, **71**, 38.
Talukdar, B., and Niyagi, K. (1982). *Physics Letters*, **119B**, 155.
Yang, C. N. (1947). *Physical Review*, **72**, 633.